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A model for bipolar current leakage in cell stacks with separate electrolyte loops

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A model predicting leakage current in a bipolar battery stack is presented. This model applies current balance and potential balance equations to a stack and treats the electrolyte, manifold and membrane separator as resistance elements in an electric circuit analog. This results in a set of linear difference equations with constant coefficients. Leakage currents in stacks made up of different numbers of cells are predicted and the effect of each resistance component on stack performance is investigated.

Nomenclature

C_j	$j = 1 \ldots 5$, constants in Equations 24–29, defined in the Appendix
D_j	$j = 1 \ldots 5$, constants in Equations 25–29, defined in the Appendix
E	difference operator, $\mathbf{E}f_n = f_{n+1}$
I_L	load current (A)
K	manifold current, anodic (A)
L	manifold current, cathodic (A)
N	number of cells in the bipolar stack
R_A	lateral electrolyte resistance, anodic (Ω)
R_C	lateral electrolyte resistance, cathodic (Ω)
R_{e1}	electrolyte resistance, anodic (Ω)
R_{e2}	electroylte resistance, cathodic (Ω)
R_{MA}	manifold resistance, anodic (Ω)
R_{MC}	manifold resistance, cathodic (Ω)
R_s	membrane resistance (Ω)
V_0	cell potential (V)
i_1	battery current, anodic (A)
i_2	battery current, cathodic (A)
k	leakage current, anodic (A)
l	leakage curent, cathodic (A)
r_j	$j = 1 \ldots 5$, roots of the characteristic equation, solved in the Appendix

1. Introduction

A bipolar battery consists of a stack of cells in which all electrodes, except the two at the ends of the stack, serve a dual function, i.e. as cathode of one cell and as anode of the next cell. The electrolyte is fed into the cells by inlet manifolds. Figure 1 shows schematically what a typical bipolar stack may look like. The advantage of a bipolar stack is that, due to the compact design (especially the elimination of tabs and intercell connection), both the energy density and power output can be quite high. Nevertheless, a shortcoming of this design is that a leakage current, sometimes called shunt or bypass current, flows between the electrodes via the interconnecting electrolyte paths among cells and manifolds. Problems arise because the leakage currents obviously cause a loss of battery efficiency. However, they may also cause corrosion of electrode material and undesired by-products [1]. Thus an analysis of the extent of leakage current is important for optimal stack design.

Several efforts have been devoted to the modeling and simulation of leakage current in a bipolar stack. Roušar and Cezner [2] were among the earliest to model the 'parasitic currents' by a network of resistances. Katz [3] first introduced nonlinear components into the equivalent circuit network. He used Zener diodes and resistances together as network components and found the leakage current by solving a set of N by N simultaneous linear equations, where N is the number of cells in the stack. Kuhn and Booth [1], reviewing the topic of bipolar leakage current, considered the use of exclusively pure resistance or Zener diodes network unrealistic, but modified the Zener diode approach of Katz and used it as a unit cell to calculate the leakage current. They were also the first to use a commercially available computer software package [4] to simulate the leakage current. Kaminski and Savinell [5] similarly used a pure resistive network, but applied the principles of difference calculus to solve the problem and expressed the leakage currents by series with a finite number of terms. The resulting set of 5 by 5 simultaneous equations was solved for the coefficients of the series. Szpak *et al.* [6] analyzed the leakage current in a cylindrical $Li/SOCl_2$ battery using essentially the same resistive network and solved the set of N by N equations, but the determination of the coefficients was more complicated because of the cylindrical geometry. White *et al.* [7] simplified the resistance network by combining anodic and cathodic path resistances, which are parallel to each other, into one resistance and applied Newman's BAND (J)

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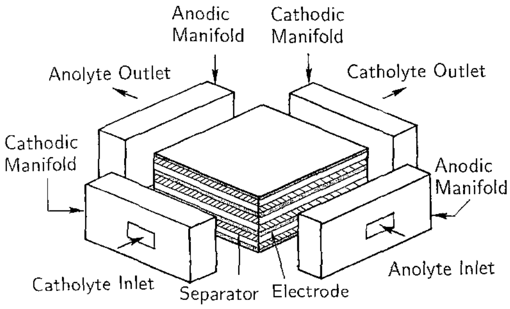

Fig. 1. Schematic of a bipolar battery stack.

subroutine [10] to solve the set of simultaneous equations.

In all cases discussed thus far, attention was focused on the method of solving for the leakage currents but not much attention was given to the relative importance of individual resistance components in the cell stack. This lack of attention is especially regrettable for the membrane separator which is an important resistance contribution. In the present paper, we extend the basic network analysis found in the literature by (1) adding a resistance component representing the separator, and (2) separating the anodic and cathodic channels. The effect of each resistance component and of the number of cells on the level of the leakage current as well as the power efficiency is investigated.

2. Theory

The bipolar battery stack is represented by the circuit diagram shown in Fig. 2, and the current component diagram is shown in Fig. 3. The load current, I_L, at the terminals of the battery is kept constant and the surfaces of the electrolyte feed system are considered nonconducting. The current distribution along the electrode is assumed uniform, so that the current paths can be represented by resistances. Cell polarization is assumed linear; it is represented by the combination of a resistance-free voltage source, V_0, and an internal resistance, which is lumped into the ohmic resistances R_{e1} or R_{e2}, of the anodic and cathodic electrolyte, respectively. (It will be demonstrated later that, as long as the total resistance in the path of the battery current is constant, the ratio of R_{e1} and R_{e2}, i.e. R_{e1}/R_{e2}, is irrelevant.)

The membrane which separates anodic and cathodic channels is represented by R_s. Electrolyte resistances in the anodic and cathodic manifold are represented by R_{MA} and R_{MC}, respectively; R_A and R_C stand for the electrolyte resistances in the anodic and cathodic channels perpendicular to the manifolds. The main current in the stack flows directly from one terminal to the other through all the cells. In each cell this 'battery current' is indicated as i^1 and i^2, on the negative and on the positive side of the separator, respectively. The leakage currents k and l in the anolyte and the catholyte channels, respectively, are assumed to be perpendicular to the battery current. They are taken to be positive when they are flowing toward the manifolds. The manifold currents K and L in the anolyte and the catholyte channels, respectively, flow parallel to the battery current (positive) or opposite to it (negative).

Applying Kirchoff's laws to this network at the nth cell analog results in a set of six simultaneous linear equations with six variables: two leakage currents (k, l), two manifold currents, (K, L) and two battery currents (i^1, i^2). The first four equations of the set are derived from the current balance equations on A, E, B and G:

$$K_{n+1} - K_n - k_{n+1} = 0 \tag{1}$$

$$L_{n+1} - L_n - l_{n+1} = 0 \tag{2}$$

$$i^1_{n+1} - i^2_n + 2k_{n+1} = 0 \tag{3}$$

$$i^2_n + 2l_n - i^1_n = 0 \tag{4}$$

The other two are the potential balance equations for loops ABCD and EFGH formed by the analog elements of the nth cell:

$$R_A(k_{n+1} - k_n) - R_{MA}K_n + (R_{e1} + R_s)i^1_n + R_{e2}i^2_n - V_0 = 0 \tag{5}$$

$$R_C(l_{n+1} - l_n) - R_{MC}L_n + (R_{e1} + R_s)i^1_{n+1} + R_{e2}i^2_n - V_0 = 0 \tag{6}$$

Equations 1–6 may be rewritten in the form of linear difference equations with constant coefficients [9]

$$(\mathbf{E} - 1)K - \mathbf{E}k = 0 \tag{7}$$

$$(\mathbf{E} - 1)L - \mathbf{E}l = 0 \tag{8}$$

$$\mathbf{E}i^1 - i^2 + 2\mathbf{E}k = 0 \tag{9}$$

$$i^2 - 2l - i^1 = 0 \tag{10}$$

$$R_A(\mathbf{E} - 1)k - R_{MA}K + (R_{e1} + R_s)i^1 + R_{e2}i^2 - V_0 = 0 \tag{11}$$

$$R_C(\mathbf{E} - 1)l - R_{MC}L + (R_{e1} + R_s)\mathbf{E}i^1 + R_{e2}i^2 - V_0 = 0 \tag{12}$$

where

$$K = K_n, \quad k = k_n, \quad L = L_n, \quad l = l_n, \quad i^1 = i^1_n, \quad i^2 = i^2_n.$$

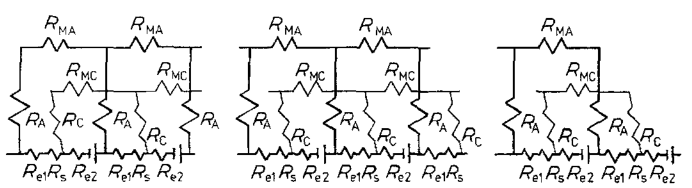

Fig. 2. Circuit analog of an N-cell assembly. For simplicity, only the upper half of the circuit is shown.

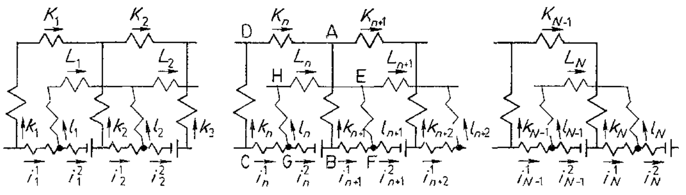

Fig. 3. Current components and node points used in Equations 1–6.

E is the difference operator:

$$\mathbf{E}f(x)/(x) = f(x+1),$$

and

$$\mathbf{E}^n f(x) = f(x+n).$$

i^2, however, may be eliminated in the above equations by the use of Equation 10. The result is a set of five linear difference equations in five variables with constant coefficients:

$$(\mathbf{E}-1)K - \mathbf{E}k = 0 \quad (13)$$

$$(\mathbf{E}-1)L - \mathbf{E}l = 0 \quad (14)$$

$$(\mathbf{E}-1)i^1 + 2l + 2\mathbf{E}k = 0 \quad (15)$$

$$R_A(\mathbf{E}-1)k - R_{MA}K + (R_{e1}+R_s)i^1 + R_{e2}(i^1 - 2l) - V_0 = 0 \quad (16)$$

$$R_C(\mathbf{E}-1)l - R_{MC}L + (R_{e1}+R_s)\mathbf{E}i^1 + R_{e2}(i^1 - 2l) - V_0 = 0 \quad (17)$$

where K, L, k, l, and i^1 are to be determined. The boundary conditions may be written as:

$$K_0 = 0 \quad (18)$$

$$K_N = 0 \quad (19)$$

$$L_0 = 0 \quad (20)$$

$$L_N = 0 \quad (21)$$

$$i_1^1 + 2k_1 = I_L \quad (22)$$

The set of Equations 13–22 may be solved by the calculus of difference equations [8]. The details of this solution are presented in the Appendix for reference. The general solution for the currents may be expressed in the form of series with a finite number of terms having constant coefficients:

$$k_n = \sum_{j=1}^{4} C_j r_j^n \quad (23)$$

$$l_n = \sum_{j=1}^{4} C_j D_j r_j^n \quad (24)$$

$$i_n^1 = -2\sum_{j=1}^{4} C_j(r_j + D_j)\frac{r_j^n}{r_j - 1} + C_5 \quad (25)$$

$$i_n^2 = -2\sum_{j=1}^{4} C_j(1 + D_j)\frac{r_j^{n+1}}{r_j - 1} + C_5 \quad (26)$$

$$K_n = \sum_{j=1}^{4}\frac{C_j r_j^{n+1}}{r_j - 1} + \frac{R_{e1}+R_{e2}+R_s}{R_{MA}}C_5 - \frac{V_0}{R_{MA}} \quad (27)$$

$$L_n = \sum_{j=1}^{4}\frac{C_j D_j r_j^{n+1}}{r_j - 1} + \frac{R_{e1}+R_{e2}+R_s}{R_{MC}}C_5 - \frac{V_0}{R_{MC}} \quad (28)$$

The definitions of the constants C_j and D_j, and the roots r_j, can be found in the Appendix.

3. Results and discussion

A systematic study was made of the effect of varying each resistance component shown in Fig. 2. The effect of varying the number of cells was also investigated. In order to compare these effects on a common basis, the leakage currents, manifold currents and battery currents were calculated for a standard case whose parameter values are listed in Table 1. Subsequently, the values of most parameters were systematically varied. The cell potential V_0 and the output load current I_L were kept constant. The total internal resistance, R_t, defined by

$$R_t = R_s + R_{e1} + R_{e2} \quad (29)$$

was constrained according to

$$R_t < \frac{V_0}{I_L} \quad (30)$$

i.e. the current produced in each cell was always greater than the load current.

Since we are interested in the relative effects of various resistance components, the choice of standard values is arbitrary. The values shown in Table 1 are generally of the correct order of magnitude, but illustrate a case of rather severe leakage current effect.

Using the values in Table 1, we calculate the leakage current, battery current and manifold current, as shown in Fig. 4. Table 2 lists their maximum values. The current in the manifold flows in the direction opposite to the battery current. The magnitude of the manifold current is zero at both ends of the stack and increases toward the middle of the stack. Accordingly, the leakage current flows away from the cell to join the manifold current in the positive half of the stack (between the positive end and the middle of the stack) and flows back into the cell in the other half. The

Table 1. Standard parameters used in the illustration

$R_A = 1000\,\Omega$	$R_C = 1000\,\Omega$
$R_{e1} = R_{e2} = 0.5\,\Omega$	$R_s = 1\,\Omega$
$R_{MA} = 10\,\Omega$	$R_{MC} = 10\,\Omega$
$V_0 = 1.5\,\text{V}$	$I_L = 0.5\,\text{A}$
$N = 20$	

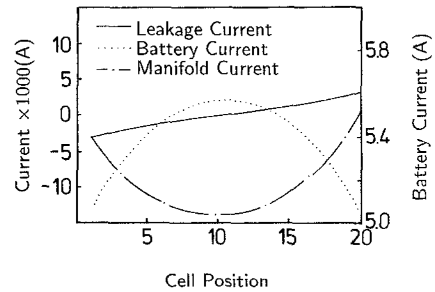

Fig. 4. Leakage current, battery current and manifold current calculated using the standard parameters of Table 1.

battery current inside the stack is always greater than the load current at the terminals and forms a maximum at the middle of the stack. As the leakage current increases, the maxima of both the manifold current and the battery current increase. This is due to the constraint that the sum of manifold current and battery current is constant. The manifold current is actually the sum of the leakage currents:

$$i_i^1 + K_i = \text{constant} \quad (31)$$

$$\vec{K} = -\sum_{n=1}^{N} \vec{k}_n \quad (32)$$

The power efficiency, defined as the ratio of power produced between the positive and negative terminals of the battery stack to the total power generated by the individual cells within the stack, is calculated according to

$$\varepsilon = \frac{\Delta V I_L}{\Sigma\, i^2 V_0} \quad (33)$$

where ΔV is the potential difference between the battery terminals and is calculated by

$$\Delta V = -k_1 R_A - \sum_{j=1}^{N-1} K_j R_{MA} + k_N R_A - (R_{e1} + R_s) i_N^1 - R_{e2} i_N^2 + V_0 \quad (34)$$

In this case, ε has the value of 26.1%. As mentioned before, the choice of the standard values is arbitrary, and in this case the low manifold resistance and high separator resistance are mainly responsible for the low efficiency.

The distribution profiles of the leakage current, battery current and the manifold current are similar in all calculations. Therefore, in the following discussion we assess the leakage effect by comparing the changes in maximum leakage current and in power efficiency.

3.1. *Number of cells*

As shown in Fig. 5, the leakage current increases as the number of cells, N, increases, but flattens out beyond 25 cells. The efficiency, on the other hand, decreases as N increases. Note that in these calculations the load current is kept constant. Thus, to maintain current output as N increases, extra current must be generated in the cells to compensate for the increase leak to the manifold. Consequently, although the battery voltage increases, the efficiency keeps decreasing with increasing N.

Table 2. Calculated results using the standard parameters

Maximum leakage current	3.032×10^{-3} A
Maximum battery current	0.556 A
Maximum manifold current	-1.418×10^{-2} A
Power efficiency	0.261

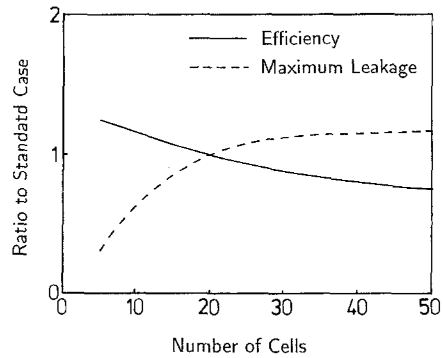

Fig. 5. Effect of the number of cells on the maximum leakage current and power efficiency in a bipolar battery stack, expressed in ratio to the values for the standard case.

3.2. *Total internal resistance*

The effect of the total internal resistance on the performance of the battery is shown in Fig. 6. The asymmetry of the electrolyte resistances R_{e1} and R_{e2} does not have much effect on the current distribution. As long as the total resistance R_t is kept constant, the values of leakage current and power efficiency vary no more than 1%, even though the ratio of R_{e1} to R_{e2} varies from 1/3 to 4 and the separator resistance from zero to 80% of the total resistance. This is expected because the lateral resistances R_A and R_C are much larger than the electrolyte resistances R_{e1} and R_{e2}.

On the other hand, if the total resistance increases, the leakage current and the power efficiency both decrease, as shown in Fig. 6. The decrease of leakage current is a result of Ohm's Law: as the total resistance is increased, the current produced in each cell by a constant voltage source is less, consequently less current is leaked to the manifolds in order to maintain a constant current ouput. The efficiency decreases

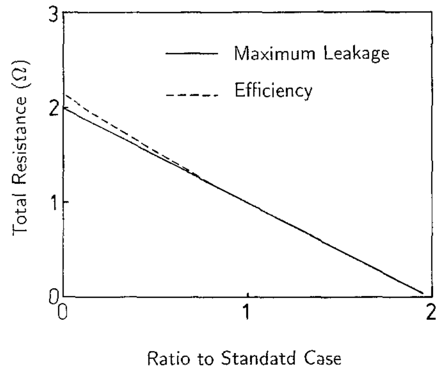

Fig. 6. Effect of total resistance on the maximum leakage current and power efficiency in a bipolar battery stack, expressed in ratio to the values for the standard case.

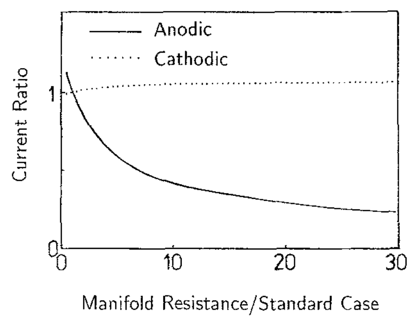

Fig. 7. Effect of anodic manifold resistance on the maximum leakage currents in a bipolar battery stack, expressed in ratio to the values for the standard case.

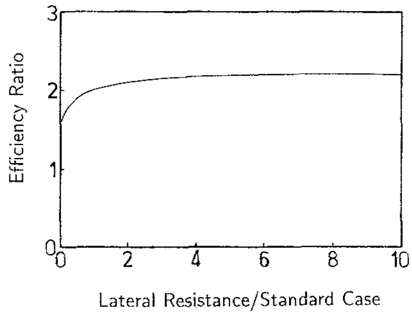

Fig. 10. Effect of lateral electrolyte resistances on the power efficiency in a bipolar battery stack, expressed in ratio to the values for the standard case.

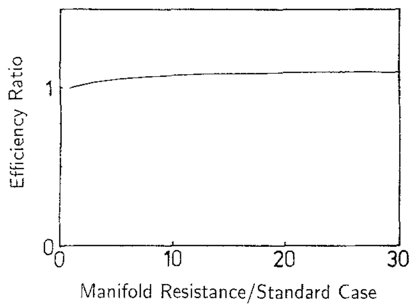

Fig. 8. Effect of manifold resistances on the power efficiency of a bipolar battery stack, expressed in ratio to the values for the standard case.

because more power is consumed to overcome the internal resistance.

3.3. *Manifold resistances*

Variation of manifold resistances affects only the compartment to which the manifold is connected, i.e. cathodic currents are affected only by the cathodic manifold resistance, but not by the anodic manifold resistance (Fig. 7). The anodic leakage current, on the other hand, decreases sharply near the symmetry point, i.e. the intersection point of the two curves in Fig. 7, where anodic and cathodic manifold resistances are the same, and tends toward a constant value.

Figure 8 shows that manifold resistance does not have much effect on the battery efficiency except near the symmetry point. This is due to the fact that the total internal resistance is constant: when the manifold resistance becomes much greater than the cell internal resistance, this will not change the potential and current distribution within the battery. Consequently, the power efficiency remains the same.

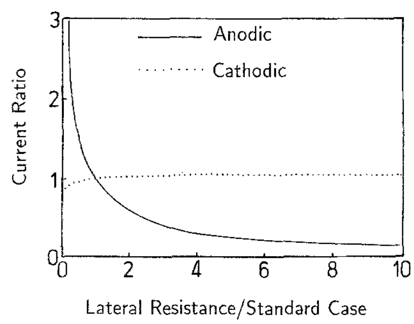

Fig. 9. Effect of anodic lateral electrolyte resistance on the maximum leakage currents in a bipolar battery stack, expressed in ratio to the values for the standard case.

3.4. *Lateral electrolyte resistances*

The effect of the electrolyte resistances in the lateral direction is similar to the effect of the manifold resistance discussed above (Figs 9 and 10). Again, R_A and R_C are much larger than the internal resistance of the cell, so the change of electrolyte resistance in the lateral direction has little effect.

4. Conclusions

(1) Calculating leakage current in a bipolar battery stack by methods of difference calculus is very efficient compared to the matrix method, in which N by N simultaneous equations are to be solved.

(2) Leakage current in a bipolar stack is most sensitive to the internal resistance in the direction of the battery current.

(3) Lateral electrolyte resistances and manifold resistances have some effect on the leakage current, but only on the leakage current in the compartment whose resistance is varied. This effect is much weaker than that of total internal resistance.

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Appendix

The governing Equations 13–17 are the result of current and voltage balances on the nth cell assembly. After eliminating i^2 by Equation 10, Equations 13–17 become:

$$(\mathbf{E} - 1)K - \mathbf{E}k = 0 \tag{13}$$

$$(\mathbf{E} - 1)L - \mathbf{E}l = 0 \tag{14}$$

$$(\mathbf{E} - 1)i^1 + 2l + 2\mathbf{E}k = 0 \tag{15}$$

$$R_A(\mathbf{E} - 1)k - R_{MA}K + (R_{e1} + R_s)i^1 + R_{e2}(i^1 - 2l) - V_0 = 0 \quad (16)$$

$$R_C(\mathbf{E} - 1)l - R_{MC}L + (R_{e1} + R_s)\mathbf{E}i^1 + R_{e2}(i^1 - 2l) - V_0 = 0 \quad (17)$$

with boundary conditions

$$K_0 = 0 \quad (18)$$

$$K_N = 0 \quad (19)$$

$$L_0 = 0 \quad (20)$$

$$L_N = 0 \quad (21)$$

$$i_1^1 + 2k_1 = I_L \quad (22)$$

K and L are eliminated from Equations 16 and 17 by Equations 13 and 14, respectively. Equations 16 and 17 then become:

$$[R_A(\mathbf{E} - 1)^2 - (2R_{e1} + 2R_{e2} + 2R_s + R_{MA})\mathbf{E}]k - [2(R_{e1} + R_{e2}) + 2R_{e2}\mathbf{E}]l = 0 \quad (35)$$

$$-2[(R_{e1} + R_s)\mathbf{E} + R_{e2}]\mathbf{E}k + [R_C(\mathbf{E} - 1)^2 - (2R_{e1} + 2R_{e2} + 2R_s + R_{MC})\mathbf{E}]l = 0 \quad (36)$$

l is further eliminated from Equations 35 and 36. The result is

$$[R_A(\mathbf{E} - 1)^2 - (2R_{e1} + 2R_{e2} + 2R_s + R_{MA})\mathbf{E}] \times [R_C(\mathbf{E} - 1)^2 - (2R_{e1} + 2R_{e2} + 2R_s + R_{MC})\mathbf{E}]k - 4[R_{e1} + R_s)\mathbf{E}^2 + R_{e2}\mathbf{E}] \times [R_{e2}\mathbf{E} + (R_{e1} + R_s)]k = 0 \quad (37)$$

Equation 37 is a linear homogeneous difference equation with constant coefficients. The order of the equation is equal to the highest power of E, i.e. 4. This is also equal to the number of the arbitrary constants that will show up in the solution [8]. If the characteristic equation has no multiple roots, the general solution of Equation 37 is

$$k = \sum_{j=1}^{4} C_j r_j^n \quad (38)$$

and the general solution for l is

$$l = \sum_{j=1}^{4} C_j D_j r_j^n \quad (39)$$

In the above equations, the quantities C_j are arbitrary independent constants to be determined from the boundary conditions, the quantities r_j are roots of the characteristic equation, and the quantities D_j are constants and functions of r_j. They are solved for after the general solutions have been found.

Substituting the solutions for k, Equation 38, into Equation 13, we arrive at the following equation for K:

$$(\mathbf{E} - 1)K = \sum_{j=1}^{4} C_j r_j^{n+1} \quad (40)$$

This is nonhomogeneous linear difference equation. We have to find both the homogeneous and particular solution to Equation 40. First, the homogeneous solution is found by substituting $r = 1$, the root of the characteristic equation $r - 1 = 0$ corresponding to Equation 40, into the above equation. The homogeneous solution $\bar{K}$ is

$$\bar{K} = C_5' r^n = C_5' \quad (41)$$

and the particular solution can be written as [8]:

$$K^* = \frac{\Sigma_{j=1}^4 C_j r_j^{n+1}}{\Sigma_{j=1}^4 r_j - 1} \quad (42)$$

Therefore the complete solution is

$$K = \bar{K} + K^* = C_5' + \sum_{j=1}^{4} \frac{C_j r_j^{n+1}}{r_j - 1} \quad (43)$$

L is obtained in a similar way:

$$L = C_5'' + \sum_{j=1}^{4} \frac{C_j D_j r_j^{n+1}}{r_j - 1} \quad (44)$$

The general solution of i^1 is obtained by substituting Equations 38 and 39 into Equation 15. The result is:

$$(E - 1)i^1 = -2\left[\sum_{j=1}^{4} C_j D_j r_j^n + \sum_{j=1}^{4} C_j r_j^{n+1}\right] \quad (45)$$

Again, this is a nonhomogeneous linear difference equation. The general solution is obtained in a similar way as K and L:

$$i^1 = C_5 - 2\sum_{j=1}^{4} C_j(D_j + r_j)\frac{r_j^n}{r_j - 1} \quad (46)$$

Substituting Equation 46 into Equation 10, we get

$$i^2 = i^1 - 2l = C_5 - 2\sum_{j=1}^{4} C_j(1 + D_j)\frac{r_j^{n+1}}{r_j - 1} \quad (47)$$

In Equations 40, 44 and 46, C_i, C_i', and C_i'' are all constants, however, C_5, C_5' and C_5'' are not independent of each other. We will find their relation as follows. Substituting K, k, L, l and i^1 into Equations 16 and 17, we obtain

$$R_A \sum_{j=1}^{4} C_j r_j^{n+1} - R_A \sum_{j=1}^{4} C_j r_j^n - R_{MA} C_5' - R_{MA} \sum_{j=1}^{4} \frac{C_j r_j^{n+1}}{r_j - 1} + (R_{e1} + R_{e2} + R_s)C_5 - 2(R_{e1} + R_{e2} + R_s)\sum_{j=1}^{4} C_j(D_j + r_j)\frac{r_j^n}{r_j - 1} - 2R_{e2}\sum_{j=1}^{4} C_j D_j r_j^n - V_0 \equiv 0 \quad (48)$$

$$R_C \sum_{j=1}^{4} C_j D_j r_j^{n+1} - R_C \sum_{j=1}^{4} C_j D_j r_j^n - R_{MC} C_5'' - R_{MC} \sum_{j=1}^{4} \frac{C_j D_j r_j^{n+1}}{r_j - 1} + (R_{e1} + R_{e2} + R_s)C_5 - 2(R_{e1} + R_{e2} + R_s)\sum_{j=1}^{4} C_j(D_j + r_j)\frac{r_j^{n+1}}{r_j - 1} - 2R_{e2}\sum_{j=1}^{4} C_j(D_j + 1)\frac{r_j^{n+1}}{r_j - 1} - V_0 \equiv 0 \quad (49)$$

Equations 48 and 49 are identical, therefore the sum of all terms with identical exponents as well as the constant terms must be zero. Thus we obtain the following relationships:

$$R_A r_j - R_A - R_{MA}\frac{r_j}{r_j - 1} - 2(R_{e1} + R_{e2} + R_s)\frac{D_j + r_j}{r_j - 1} - 2R_{e2}D_j = 0$$

$$-R_{MA}C_5' + (R_{e1} + R_{e2} + R_s)C_5 - V_0 = 0$$

$$-R_{MC}C_5'' + (R_{e1} + R_{e2} + R_s)C_5 - V_0 = 0$$

Solving these for C_5', C_5'' and D_j in terms of C_5 and r_j:

$$C_5' = \frac{R_{e1} + R_{e2} + R_s}{R_{MA}}C_5 - \frac{V_0}{R_{MA}} \tag{50}$$

$$C_5'' = \frac{R_{e1} + R_{e2} + R_s}{R_{MC}}C_5 - \frac{V_0}{R_{MC}} \tag{51}$$

$$D_j = \frac{R_A(r_j - 1)^2 - (2R_{e1} + 2R_{e2} + 2R_s + R_{MA})r_j}{2(R_{e1} + R_s) + 2R_{e2}r_j} \tag{52}$$

Using the above equations, we may then rewrite the general solutions (Equations 23–28) as:

$$k_n = \sum_{j=1}^{4} C_j r_j^n$$

$$l_n = \sum_{j=1}^{4} C_j D_j r_j^n$$

$$i_n^1 = -2\sum_{j=1}^{4} C_j(r_j + D_j)\frac{r_j^n}{r_j - 1} + C_5$$

$$i_n^2 = -2\sum_{j=1}^{4} C_j(1 + D_j)\frac{r_j^{n+1}}{r_j - 1} + C_5$$

$$K_n = \sum_{j=1}^{4}\frac{C_j r_j^{n+1}}{r_j - 1} + \frac{R_{e1} + R_{e2} + R_s}{R_{MA}}C_5 - \frac{V_0}{R_{MA}}$$

$$L_n = \sum_{j=1}^{4}\frac{C_j D_j r_j^{n+1}}{r_j - 1} + \frac{R_{e1} + R_{e2} + R_s}{R_{MC}}C_5 - \frac{V_0}{R_{MC}}$$

The quantities r_j ($j = 1 \ldots 4$), as mentioned above, are the roots of the characteristic equation of the system, Equation 37, and can now be evaluated. Rewriting Equation 37 with the aid of Equation 38:

$$\begin{aligned}&\{R_A R_C \mathbf{E}^4 + [-4R_A R_C - R_C(2R_{e1} + 2R_{e2} + 2R_s + R_{MA}) - R_A(2R_{e1} + 2R_{e2} + 2R_s + R_{MC}) - 4R_{e2}(R_{e1} + R_s)]\mathbf{E}^3\\ &+ [6R_A R_C + 2R_C(2R_{e1} + 2R_{e2} + 2R_s + R_{MA}) + 2R_A(2R_{e1} + R_{e2} + 2R_s + R_{MA})\\ &+ (2R_{e1} + R_{e2} + 2R_s + R_{MC})(2R_{e1} + 2R_{e2} + 2R_s + R_{MC}) - 4R_{e2}^2 - 4(R_{e1} + R_s)^2]\mathbf{E}^2\\ &+ [-4R_A R_C - R_C(2R_{e1} + 2R_{e2} + 2R_s + R_{MA}) - R_A(2R_{e1} + 2R_{e2} + 2R_s + R_{MC})\\ &- 4R_{e2}(R_{e1} + R_s)]\mathbf{E} + R_A R_C\}k = 0\end{aligned} \tag{53}$$

If we define

$$\begin{aligned}A \equiv{}& 4R_A R_C + R_C(2R_{e1} + 2R_{e2} + 2R_s + R_{MA})\\ &+ R_A(2R_{e1} + 2R_{e2} + 2R_s + R_{MC})\\ &+ 4R_{e2}(R_{e1} + R_s)\end{aligned} \tag{54}$$

$$\begin{aligned}B \equiv{}& 6R_A R_C + 2R_C(2R_{e1} + 2R_{e2} + 2R_s + R_{MA})\\ &+ 2R_A(2R_{e1} + 2R_{e2} + 2R_s + R_{MC})\\ &+ (2R_{e1} + 2R_{e2} + R_s + R_{MA})(2R_{e1} + 2R_{e2}\\ &+ 2R_s + R_{MC}) - 4R_{e2}^2 - 4(R_{e1} + R_s)^2\end{aligned} \tag{55}$$

$$C \equiv R_A R_C \tag{56}$$

and let r be the root of Equation 53, then the latter becomes

$$Cr^4 - Ar^3 + Br^2 - Ar + C = 0$$

$$(r^2 - ar + 1)(r^2 - br + 1) = 0$$

Solving for r, we obtain:

$$r_{1,2} = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - 1} \tag{57}$$

$$r_{3,4} = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - 1} \tag{58}$$

where $a + b = A/C$, and $ab + 2 = B/C$, a and b are defined as

$$a = \frac{1}{2}\left(\frac{A}{C} + \sqrt{\left(\frac{A}{C}\right)^2 - 4\left(\frac{B}{C} - 2\right)}\right) \tag{59}$$

$$b = \frac{1}{2}\left(\frac{A}{C} + \sqrt{\left(\frac{A}{C}\right)^2 - 4\left(\frac{B}{C} - 2\right)}\right) \tag{60}$$

Finally, we solve for C_j ($j = 1 \ldots 5$) by substituting Equations 43, 44 and 46 into the boundary conditions, Equations 18–22,

$$K_0 = \sum_{j=1}^{4} C_j\frac{r_j}{r_j - 1} + \frac{R_{e1} + R_{e2} + R_s}{R_{MA}}C_5 - \frac{V_0}{R_{MA}} = 0 \tag{61}$$

$$K_N = \sum_{j=1}^{4} C_j\frac{r_j^{N+1}}{r_j - 1} + \frac{R_{e1} + R_{e2} + R_s}{R_{MA}}C_5 - \frac{V_0}{R_{MA}} = 0 \tag{62}$$

$$L_0 = \sum_{j=1}^{4} C_j D_j\frac{r_j}{r_j - 1} + \frac{R_{e1} + R_{e2} + R_s}{R_{MC}}C_5 - \frac{V_0}{R_{MC}} = 0 \tag{63}$$

$$L_N = \sum_{j=1}^{4} C_j D_j\frac{r_j^{N+1}}{r_j - 1} + \frac{R_{e1} + R_{e2} + R_s}{R_{MC}}C_5 - \frac{V_0}{R_{MC}} = 0 \tag{64}$$

$$i_1^1 + 2k_1 = 2\sum_{j=1}^{4} C_j\left(1 - \frac{r_j + D_j}{r_j - 1}\right) + C_5 = I_L \tag{65}$$

where r_j, D_j, $(j = 1, \ldots 4)$, R_e, R_s, R_{MA}, R_{MC}, and V_0 are all known. The constant coefficients C_j are easily solved for by inverting a 5×5 matrix according to Equations 61–55. Once the coefficients C_j are known, the leakage current can be calculated immediately.

Note: It is obvious that the coefficients C_j can be solved if, and only if, $r_j \neq 1$. If we assume $r = 1$ and enter this into either Equation 57 or 58, we will get, with the help of Equations 59 and 60,

$$2\frac{A}{C} - \frac{B}{C} - 2 = 0$$

Substituting the definitions of A, B and C into the above equation, it becomes

$$\frac{2R_{MA}(R_{e1} + R_{e2} + R_s)}{R_A R_C} + \frac{2R_{MC}(R_{e1} + R_{e2} + R_s)}{R_A R_C} \frac{R_{MA} R_{MC}}{R_A R_C} = 0 \quad (66)$$

When the combination of the resistances happens to satisfy Equation 66, we may have difficulty in calculating the coefficients C_j because it involves the inversion of a singular matrix.

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